

## X GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN THE CORRESPONDENCE ROUND

Below is the list of problems for the first (correspondence) round of the X Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four elder grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem or of each its item if there are any, costs 7 points. An incomplete solution costs from 1 to 6 points according to the extent of advancement. If no significant advancement was achieved, the mark is 0. The result of a participant is the total sum of marks for all problems.

**In your work, please start the solution for each problem in a new page.** First write down the statement of the problem, and then the solution. Present your solutions in detail, including all significant arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

**The solutions for the problems (in Russian or in English) must be delivered up to April 1, 2014.** For this, please apply **since January 2, 2014** to <http://olimpsharygin.olimpiada.ru> and follow the instructions given there. **Attention:** the solution of each problem must be contained in a separate pdf, doc or jpg file. We recommend to prepare the paper using a computer or to scan it rather than to photograph it. In the last two cases, please check readability of the file obtained.

If you have some technical problem, please contact us by e-mail [geomolymp@mccme.ru](mailto:geomolymp@mccme.ru).

It is also possible to send the solutions by e-mail to [geompapers@yandex.ru](mailto:geompapers@yandex.ru). In this case, please follow a few simple rules:

1. *Each student sends his work in a separate message (with delivery notification). The size of the message must not exceed 10 Mb.*

2. *If your work consists of several files, send it as an archive.*

3. *If the size of your message exceeds 10 Mb, divide it into several messages.*

4. *In the subject of the message write "The work for Sharygin olympiad", and present the following personal data in the body of your message:*

- last name;
- all other names;
- E-mail, phone number, post address;
- the current number of your grade at school;
- the last grade at your high school;
- the number of the last grade in your school system;
- the number and/or the name and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no possibility to deliver the work in electronic form, please apply to the Organizing Committee to find a specific solution for this case.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round in Summer 2014 in the city of Dubna, in Moscow region. (For instance, if the last grade is 12, then we invite winners from 9, 10, and 11 grade.) Winners of the correspondence round, the students of the last grade, will be awarded with diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2014. If you want to know your detailed results, please contact us by e-mail **geomolymp@mccme.ru**.

1. (8) A right-angled triangle  $ABC$  is given. Its cathetus  $AB$  is the base of a regular triangle  $ADB$  lying in the exterior of  $ABC$ , and its hypotenuse  $AC$  is the base of a regular triangle  $AEC$  lying in the interior of  $ABC$ . Lines  $DE$  and  $AB$  meet at point  $M$ . The whole configuration except points  $A$  and  $B$  was erased. Restore the point  $M$ .
2. (8) A paper square with sidelength 2 is given. From this square, can we cut out a 12-gon having all sidelengths equal to 1, and all angles divisible by  $45^\circ$ ?
3. (8) Let  $ABC$  be an isosceles triangle with base  $AB$ . Line  $\ell$  touches its circumcircle at point  $B$ . Let  $CD$  be a perpendicular from  $C$  to  $\ell$ , and  $AE$ ,  $BF$  be the altitudes of  $ABC$ . Prove that  $D$ ,  $E$ ,  $F$  are collinear.
4. (8) A square is inscribed into a triangle (one side of the triangle contains two vertices and each of two remaining sides contains one vertex). Prove that the incenter of the triangle lies inside the square.
5. (8) In an acute-angled triangle  $ABC$ ,  $AM$  is a median,  $AL$  is a bisector and  $AH$  is an altitude ( $H$  lies between  $L$  and  $B$ ). It is known that  $ML = LH = HB$ . Find the ratios of the sidelengths of  $ABC$ .
6. (8–9) Given a circle with center  $O$  and a point  $P$  not lying on it. Let  $X$  be an arbitrary point of this circle, and  $Y$  be a common point of the bisector of angle  $POX$  and the perpendicular bisector to segment  $PX$ . Find the locus of points  $Y$ .
7. (8–9) A parallelogram  $ABCD$  is given. The perpendicular from  $C$  to  $CD$  meets the perpendicular from  $A$  to  $BD$  at point  $F$ , and the perpendicular from  $B$  to  $AB$  meets the perpendicular bisector to  $AC$  at point  $E$ . Find the ratio in which side  $BC$  divides segment  $EF$ .
8. (8–9) Given a rectangle  $ABCD$ . Two perpendicular lines pass through point  $B$ . One of them meets segment  $AD$  at point  $K$ , and the second one meets the extension of side  $CD$  at point  $L$ . Let  $F$  be the common point of  $KL$  and  $AC$ . Prove that  $BF \perp KL$ .
9. (8–9) Two circles  $\omega_1$  and  $\omega_2$  touching externally at point  $L$  are inscribed into angle  $BAC$ . Circle  $\omega_1$  touches ray  $AB$  at point  $E$ , and circle  $\omega_2$  touches ray  $AC$  at point  $M$ . Line  $EL$  meets  $\omega_2$  for the second time at point  $Q$ . Prove that  $MQ \parallel AL$ .

10. (8–9) Two disjoint circles  $\omega_1$  and  $\omega_2$  are inscribed into an angle. Consider all pairs of parallel lines  $l_1$  and  $l_2$  such that  $l_1$  touches  $\omega_1$ , and  $l_2$  touches  $\omega_2$  ( $\omega_1, \omega_2$  lie between  $l_1$  and  $l_2$ ). Prove that the medial lines of all trapezoids formed by  $l_1, l_2$  and the sides of the angle touch some fixed circle.
11. (8–9) Points  $K, L, M$  and  $N$  lying on the sides  $AB, BC, CD$  and  $DA$  of a square  $ABCD$  are vertices of another square. Lines  $DK$  and  $NM$  meet at point  $E$ , and lines  $KC$  and  $LM$  meet at point  $F$ . Prove that  $EF \parallel AB$ .
12. (9–10) Circles  $\omega_1$  and  $\omega_2$  meet at points  $A$  and  $B$ . Let points  $K_1$  and  $K_2$  of  $\omega_1$  and  $\omega_2$  respectively be such that  $K_1A$  touches  $\omega_2$ , and  $K_2A$  touches  $\omega_1$ . The circumcircle of triangle  $K_1BK_2$  meets lines  $AK_1$  and  $AK_2$  for the second time at points  $L_1$  and  $L_2$  respectively. Prove that  $L_1$  and  $L_2$  are equidistant from line  $AB$ .
13. (9–10) Let  $AC$  be a fixed chord of a circle  $\omega$  with center  $O$ . Point  $B$  moves along the arc  $AC$ . A fixed point  $P$  lies on  $AC$ . The line passing through  $P$  and parallel to  $AO$  meets  $BA$  at point  $A_1$ ; the line passing through  $P$  and parallel to  $CO$  meets  $BC$  at point  $C_1$ . Prove that the circumcenter of triangle  $A_1BC_1$  moves along a straight line.
14. (9–11) In a given disc, construct a subset such that its area equals the half of the disc area and its intersection with its reflection over an arbitrary diameter has the area equal to the quarter of the disc area.
15. (9–11) Let  $ABC$  be a non-isosceles triangle. The altitude from  $A$ , the bisector from  $B$  and the median from  $C$  concur at point  $K$ .
  - a) Which of the sidelengths of the triangle is medial?
  - b) Which of the lengths of segments  $AK, BK, CK$  is medial?
16. (9–11) Given a triangle  $ABC$  and an arbitrary point  $D$ . The lines passing through  $D$  and perpendicular to segments  $DA, DB, DC$  meet lines  $BC, AC, AB$  at points  $A_1, B_1, C_1$  respectively. Prove that the midpoints of segments  $AA_1, BB_1, CC_1$  are collinear.
17. (10–11) Let  $AC$  be the hypotenuse of a right-angled triangle  $ABC$ . The bisector  $BD$  is given, and the midpoints  $E$  and  $F$  of the arcs  $BD$  of the circumcircles of triangles  $ADB$  and  $CDB$  respectively are marked (the circles are erased). Construct the centers of these circles using only a ruler.
18. (10–11) Let  $I$  be the incenter of a circumscribed quadrilateral  $ABCD$ . The tangents to circle  $AIC$  at points  $A, C$  meet at point  $X$ . The tangents to circle  $BID$  at points  $B, D$  meet at point  $Y$ . Prove that  $X, I, Y$  are collinear.
19. (10–11) Two circles  $\omega_1$  and  $\omega_2$  touch externally at point  $P$ . Let  $A$  be a point of  $\omega_2$  not lying on the line through the centers of the circles, and  $AB, AC$  be the tangents to  $\omega_1$ . Lines  $BP, CP$  meet  $\omega_2$  for the second time at points  $E$  and  $F$ . Prove that line  $EF$ , the tangent to  $\omega_2$  at point  $A$  and the common tangent at  $P$  concur.
20. (10–11) A quadrilateral  $KLMN$  is given. A circle with center  $O$  meets its side  $KL$  at points  $A$  and  $A_1$ , side  $LM$  at points  $B$  and  $B_1$ , etc. Prove that if the circumcircles of triangles  $KDA, LAB, MBC$  and  $NCD$  concur at point  $P$ , then

- a) the circumcircles of triangles  $KD_1A_1$ ,  $LA_1B_1$ ,  $MB_1C_1$  and  $NC_1D_1$  also concur at some point  $Q$ ;
- b) point  $O$  lies on the perpendicular bisector to  $PQ$ .
21. (10–11) Let  $ABCD$  be a circumscribed quadrilateral. Its incircle  $\omega$  touches sides  $BC$  and  $DA$  at points  $E$  and  $F$  respectively. It is known that lines  $AB$ ,  $FE$  and  $CD$  concur. The circumcircles of triangles  $AED$  and  $BFC$  meet  $\omega$  for the second time at points  $E_1$  and  $F_1$ . Prove that  $EF \parallel E_1F_1$ .
22. (10–11) Does there exist a convex polyhedron such that it has diagonals and each of them is shorter than each of its edges?
23. (11) Let  $A$ ,  $B$ ,  $C$  and  $D$  be a triharmonic quadruple of points, i.e

$$AB \cdot CD = AC \cdot BD = AD \cdot BC.$$

Let  $A_1$  be a point distinct from  $A$  such that the quadruple  $A_1$ ,  $B$ ,  $C$  and  $D$  is triharmonic. Points  $B_1$ ,  $C_1$  and  $D_1$  are defined similarly. Prove that

- a)  $A$ ,  $B$ ,  $C_1$ ,  $D_1$  are concyclic;
- b) the quadruple  $A_1$ ,  $B_1$ ,  $C_1$ ,  $D_1$  is triharmonic.
24. (11) A circumscribed pyramid  $ABCD S$  is given. The opposite sidelines of its base meet at points  $P$  and  $Q$  in such a way that  $A$  and  $B$  lie on segments  $PD$  and  $PC$  respectively. The inscribed sphere touches faces  $ABS$  and  $BCS$  at points  $K$  and  $L$ . Prove that if  $PK$  and  $QL$  are coplanar then the touching point of the sphere with the base lies on  $BD$ .